Midterm - Linear Algebra I (2025-26)
Time: 2.5 hours.
Attempt all questions. The total marks is 30.

1. Let V be a vector space and $A \subseteq V$. Suppose $\text{Span}(A) = S$. Show that no proper subset of A generates S if and only if A is linearly independent. (recall that $A \subseteq V$ is said to be linearly independent if every finite subset of A is linearly independent) **[5 marks]**

2. Call a sequence $(a_1, a_2, \cdots)$ of real numbers is a *Fibonacci sequence* if $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. Show that the set of all Fibonacci sequences forms a vector space over $\mathbf{R}$ under component-wise additon and scalar multiplication defined in a natural way, and that it is isomorphic to $\mathbf{R}^2$. **[5 marks]**

3. Let A be an $m \times n$ matrix over $\mathbf{R}$. Let S be the subspace of $\mathbf{R}^m$ generated by the columns of A. Let $W = \{\mathbf{x} \in \mathbf{R}^n : A\mathbf{x} = \mathbf{0}\}$. Show that

$$\dim(S) + \dim\big(\text{Span}(W)\big) = n. \quad \textbf{[5 marks]}$$

4. In the vector space $\mathbf{R}^4$, find two different complements of the subspace $S = \{(x_1, x_2, x_3, x_4) : x_3 - x_4 = 0\}$. **[5 marks]**

5. Let V be a vector space over $\mathbf{R}$. Suppose $T : V \to \mathbf{R}$ is a linear transformation, and let $\text{null}(T)$ denote the null space of T. Prove that if $\mathbf{u} \in V$ is not in $\text{null}(T)$ then

$$V = \text{null}(T) \oplus \{a\mathbf{u} : a \in \mathbf{R}\}. \quad \textbf{[5 marks]}$$

6. Let T be the linear operator on $\mathbf{R}^2$ defined by

$$T(x_1, x_2) = (-x_2, x_1).$$

Find $[T]_{\mathcal{B}}$, the matrix of T in the ordered basis $\mathcal{B} = \{\mathbf{x}, \mathbf{y}\}$ where $\mathbf{x} = (1, 2)$ and $\mathbf{y} = (1, -1)$. **[5 marks]**